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Borel summation and momentum-plane analyticity in perturbative QCD

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Abstract: We derive a compact expression for the Borel sum of a QCD amplitude in terms of the inverse Mellin transform of the corresponding Borel function. The result allows us to investigate the momentum-plane analyticity properties of the Borel-summed Green functions in perturbative QCD. An interesting connection between the asymptotic behaviour of the Borel transform and the Landau singularities in the momentum plane is established. We consider for illustration the polarization function of massless quarks and the resummation of one-loop renormalon chains in the large-β_0 limit, but our conclusions have a more general validity.

Keywords: QCD, Renormalization Regularization and Renormalons.

JHEP03(1999)007

Contents

1. **Introduction** **1**

2. **Borel transforms and distribution functions** **3**

3. **Borel sums in the complex momentum plane** **9**

4. **Analyticity properties of the Borel-summed functions** **13**

5. **Conclusions** **18**

A. **Relations between distribution functions** **20**

1. Introduction

It is well-known that the momentum-plane analyticity properties imposed by causality and unitarity are not automatically satisfied by the correlation functions calculated in perturbative QCD. This is actually a more general feature of approximate solutions of physical equations. For instance, unexpected analytic properties were discovered for the perturbative solutions in potential theory [1]. In QCD, the problem is more severe due to confinement. Indeed, as shown in [2], in a confined theory the poles and branch points of the true Green functions are generated by the physical hadron states in the unitarity relation, and no singularities related to the underlying quark and gluon degrees of freedom should appear. On the other hand, in perturbation theory the branch points of the amplitudes correspond to the production of free quarks and gluons. Other, more complicated singularities make their appearance if one goes beyond naive perturbation theory. For instance, renormalization-group improved expansions have unphysical space-like singularities (Landau poles or cuts), while the introduction of non-perturbative contributions such as vacuum condensates in the Operator Product Expansion (OPE) produces poles and branch cuts at the origin [3]. Both types of singularities are absent from the exact QCD amplitudes.

Recently, the contribution of large orders of perturbation theory related to the so-called renormalons [4] have been investigated by several authors, using in particular the method of Borel summation [5]–[26] (see [27] for a review and further references). In general, these resummations introduce additional momentum-plane singularities in QCD Green functions, reflecting the presence of the ultraviolet (UV) and infrared

(IR) renormalons in their Borel transforms. But despite of the large number of works devoted to Borel summation in perturbative QCD, the momentum-plane analyticity of the Borel sum did not receive particular attention so far. The reason presumably lies in the fact that perturbation theory by itself is not complete and must be supplemented by non-perturbative terms to build up the exact theory. It is therefore not surprising that theoretical predictions based on perturbation theory are plagued by bad analyticity properties absent from the true physical amplitudes. Nevertheless, given that analyticity is the key property allowing the continuation of theoretical predictions from Euclidian to Minkowskian kinematics relevant to hadronic observables, we believe an investigation of the momentum-plane analyticity of the Borel sum in perturbative QCD is of both theoretical and practical interest.

The importance of the problem was emphasized some time ago by Khuri [28], who noted the mathematical possibility of a Borel sum with analytic and asymptotic properties in the momentum plane different from those of the individual terms in the perturbative expansion. (Khuri also quotes unpublished results by J.P. Eckmann and T. Spencer illustrating this possibility.) The deep connection between momentum-plane analyticity and Borel summability was revealed by 't Hooft [4], who showed that the presence of resonances and multi-particle thresholds on the time-like axis is in conflict with the mathematical conditions required for the existence of the Borel sum. More recently, the relation between the asymptotic behaviour of the Borel transform and the momentum-plane singularities of the QCD amplitudes was discussed in [18]. Renormalization-group invariance and the Landau pole in the running coupling $\alpha_s(Q^2)$ are the essential ingredients of this analysis. Attempts to remove the unphysical Landau singularity in the context of dispersion theory with an IR-regular running coupling were made in [29]. This procedure raised much interest, since it generates power corrections of a type not present in the OPE [30, 31] (see, however, the discussion in [32]). Further clarifications of the connection between the Landau singularity of the running coupling and the IR renormalons of the Borel transform are desirable in this context.

In the present paper, we discuss some aspects of the momentum-plane analyticity of Borel-summed Green functions in perturbative QCD. To this end, we consider the polarization function for massless quarks and calculate its Borel sum at complex values of the momentum transfer. The result we derive, written in a compact form in terms of a "distribution function" (i.e., an inverse Mellin transform of the Borel function), allows us to investigate the singularities generated by the process of Borel summation. Since we consider the massless case, branch points at $s = 0$ will be present in the time-like region, which of course do not reproduce the true hadronic thresholds. Our main concern, however, is the appearance of unphysical (Landau) singularities in the space-like region. Our discussion is kept at a rather formal level, although reference to the specific case of a single renormalon chain (corresponding to the so-called large-β_0 limit [15, 27]) is made to illustrate various results and as-

JHEP03(1999)007

sumptions. In section 2, we start with a brief review of the integral representations of the Borel function in terms of its inverse Mellin transforms. Two different types of inverse Mellin transforms (called distribution functions) were introduced in [15] and [18]. We indicate the mathematical conditions under which these functions exist, the relations among them, and their connection with renormalons. An important finding is that the inverse Mellin transform of the Borel function associated with the polarization amplitude is, in general, a piece-wise analytic function, with two pieces defined inside and outside a circle in the complex plane, each of them being analytic except for a cut on the real axis. In section 3, we derive a compact expression for the polarization function at complex values of the momentum variable in terms of the distribution function defined in [15]. The same technique is applied for obtaining an expression for the Borel sum of Minkowskian quantities, using as input the standard result for the Borel transform of these quantities obtained by analytic continuation. In section 4, we discuss the analyticity properties of the polarization amplitude in the momentum plane. These properties depend on the prescription used for treating the IR renormalon singularities of the Borel transform. We adopt a principal-value prescription, which leads to amplitudes without space-like singularities outside the "Landau region" (the region between 0 and the location of Landau singularities on the space-like s axis in the renormalization-group improved perturbative expansions of correlators in QCD) and is therefore consistent with the analyticity requirements imposed by causality and unitarity. The analytic continuation to low momenta reveals explicitly the connection between the Landau singularities and the piece-wise character of the inverse Mellin transform, which in turn is determined by the asymptotic behaviour of the Borel function. Our conclusions are given in section 5. The paper has an Appendix, in which we present a method for deriving relations between the various distribution functions based on Parseval's theorem for Mellin transforms [34].

2. Borel transforms and distribution functions

Consider the current–current correlation function $\Pi(q^2)$ defined by

$$-4\pi^2 i \int d^4x \, e^{iq\cdot x} \langle 0| T\left\{V^\mu(x), V^\nu(0)\right\} |0\rangle = (q^\mu q^\nu - g^{\mu\nu} q^2)\, \Pi(q^2)\,, \tag{2.1}$$

where $V^\mu = \bar{q}\gamma^\mu q$ is the conserved vector current for massless quarks. We define the corresponding Adler function

$$D(s) = -s\,\frac{d\Pi(s)}{ds}\,, \qquad s = q^2\,, \tag{2.2}$$

which is UV finite. The function $\Pi(s)$ can be obtained from $D(s)$ by logarithmic integration, yielding

$$\Pi(s) = -\int^{s} d\ln(-s')\, D(s') + \text{const.}\,, \tag{2.3}$$

JHEP03(1999)007

where the integration is along a contour in the complex s' plane ending at the point s and not encountering the singularities of the integrand. General causality and unitarity properties of QCD Green functions imply that $\Pi(s)$ and $D(s)$ are real analytic functions in the complex s plane (i.e., $\Pi(s^*) = \Pi^*(s)$ and $D(s^*) = D^*(s)$), cut along the positive real axis from the threshold for hadron production at $s = 4m_\pi^2$ to infinity. The analyticity properties of the exact Adler function $D(s)$ are summarized in the dispersion relation

$$D(s) = \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \, \frac{Im\,\Pi(s'+i\epsilon)}{(s'-s)^2} \,, \tag{2.4}$$

where the spectral function $Im\,\Pi(s+i\epsilon)$ is non-negative. The function $\Pi(s)$ satisfies a once-subtracted dispersion relation similar to (2.4).

Writing the one-loop running coupling constant as

$$\alpha_s(-s) = \frac{4\pi}{\beta_0 \ln(-s/\Lambda^2)} \,, \tag{2.5}$$

where $\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f$ is the first coefficient of the β function and Λ the QCD scale parameter, one finds that the first terms in the renormalization-group improved perturbative expansions of the functions $D(s)$ and $\Pi(s)$ are given by

$$\begin{aligned} D(s) &= 1 + \frac{4}{\beta_0 \ln(-s/\Lambda^2)} + \cdots \,, \\ \Pi(s) &= k - \ln(-s/\Lambda^2) - \frac{4 \ln\ln(-s/\Lambda^2)}{\beta_0} + \cdots \,, \end{aligned} \tag{2.6}$$

where k is a constant. These expressions are derived using the renormalization-group equations in the perturbative region $|s| \gg \Lambda^2$. Their analytic continuation to low energy contains unphysical singularities on the space-like axis, which are absent from the exact amplitudes. Specifically, $D(s)$ has a Landau pole at $s = -\Lambda^2$, and $\Pi(s)$ has a Landau cut along the interval $-\Lambda^2 < s < 0$. In [29] it was proposed to restore the correct analyticity properties by a redefinition of the coupling (2.5) using a dispersion relation. In the present paper, we take a different perspective and analyse the impact of large orders of perturbation theory on the analyticity properties of the correlation functions.

The structure of high-order contributions to the perturbative expansion of the Adler function was investigated by many authors. The expansion coefficients are known to exhibit a factorial growth, indicating that the perturbation series has zero radius of convergence [7, 8]. If one formally applies the Borel method to sum the series, this growth is reflected in singularities of the Borel transform on the real axis. Let us write the formal Laplace integral expressing $D(s)$ in terms of its Borel transform in the large-β_0 limit as [27]

$$
\begin{aligned}
D(s) &= 1 + \frac{1}{\beta_0}\int_0^\infty du \left(\frac{-se^{-C}}{\mu^2}\right)^{-u} \exp\left(-\frac{4\pi u}{\beta_0 \alpha_s(\mu^2)}\right) \widehat{B}_D(u) \\
&= 1 + \frac{1}{\beta_0}\int_0^\infty du \left(\frac{-s}{\Lambda_V^2}\right)^{-u} \widehat{B}_D(u)\,, \qquad (2.7)
\end{aligned}
$$

where C is a scheme-dependent constant, which takes the value $C = -5/3$ in the $\overline{\mathrm{MS}}$ scheme and $C = 0$ in the V scheme, and $\Lambda_V^2 = e^{-C}\Lambda^2$ is a scheme-independent combination of this constant with the QCD scale parameter. The corresponding representation for $\Pi(s)$ is

$$
\Pi(s) = k - \ln\left(\frac{-s}{\Lambda_V^2}\right) + \frac{1}{\beta_0}\int_0^\infty du \left(\frac{-s}{\Lambda_V^2}\right)^{-u} \widehat{B}_\Pi(u)\,. \qquad (2.8)
$$

We have used the renormalization-group invariance to isolate the dependence on the momentum variable s, so that the Borel transforms $\widehat{B}_D(u)$ and $\widehat{B}_\Pi(u)$ are scale and scheme independent and also do not depend on s. Note that the definition (2.2) implies the relation

$$
\widehat{B}_\Pi(u) = \frac{1}{u}\,\widehat{B}_D(u)\,. \qquad (2.9)
$$

The Borel transforms $\widehat{B}_D(u)$ and $\widehat{B}_\Pi(u)$ have singularities on the real axis in the complex u plane. Some of them, the so-called IR renormalons, are situated along the integration contour in (2.7) and (2.8).[1] The precise nature of these singularities is only known for some specific cases. For instance, the first IR and UV singularities are branch points located at $u = 2$ and $u = -1$, respectively, with a universal nature determined by renormalization-group coefficients [5, 26]. Information on the other singularities is only available from calculations performed in the large-β_0 limit, which predict the correct locations of the singularities but not their nature (in general, the large-β_0 approximation yields pole singularities rather than branch points). Specifically, the expression for the Borel transform of the Adler function obtained in this approximation is [7, 8]

$$
\widehat{B}_D(u) = \frac{128}{3(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k\, k}{[k^2 - (1-u)^2]^2}\,. \qquad (2.10)
$$

The presence of IR renormalons renders the perturbation series not Borel summable and the Laplace integrals ill defined. In order to evaluate these integrals, a prescription for avoiding the singularities must be specified. A question we address in this

[1] The function $\widehat{B}_\Pi(u)$ has, in addition, a pole at $u = 0$ related to the short-distance behaviour of the current correlator. This pole is removed by standard UV renormalization.

paper is whether additional information can be employed to prefer one choice of a prescription over another. We shall argue that momentum-plane analyticity might be a helpful guiding principle in this respect.

We also consider Minkowskian quantities such as the finite-energy moments of the spectral function of $\Pi(s)$ defined as

$$M_k = \frac{k+1}{\pi\, s_0^{k+1}} \int_{4m_\pi^2}^{s_0} ds\, s^k\, Im\,\Pi(s+i\epsilon) = -\frac{k+1}{2\pi i\, s_0^{k+1}} \oint_{|s|=s_0} ds\, s^k\, \Pi(s)\,, \tag{2.11}$$

where $s_0 > 4m_\pi^2$ is an arbitrary cutoff, and we have applied the Cauchy relation for the true, physical polarization function $\Pi(s)$ to pass from the integral along the cut to a contour integral along a circle in the complex s plane. Combinations of such moments enter the prediction for physical observables such as the hadronic decay rate of the τ-lepton [20], and some tests of QCD related to the dependence under variation of s_0 are discussed in [35, 36]. The Borel representation for Minkowskian quantities such as the spectral moments M_k can be derived starting from the contour integral written in the second relation in (2.11) and inserting the expression (2.8) for $\Pi(s)$. The result is

$$M_k = 1 + \frac{1}{\beta_0} \int_0^\infty du \left(\frac{s_0}{\Lambda_V^2}\right)^{-u} \widehat{B}_{M_k}(u)\,, \qquad s_0 > \Lambda_V^2\,, \tag{2.12}$$

where [20]

$$\widehat{B}_{M_k}(u) = \frac{k+1}{k+1-u}\, \frac{\sin \pi u}{\pi u}\, \widehat{B}_D(u)\,. \tag{2.13}$$

Next we define the "distribution functions" coresponding to the above Borel transforms. We consider first the Borel transform $\widehat{B}_D(u)$ of the Adler function and assume that it is analytic in the strip $u_1 < Re\, u < u_2$, where $u_1 = -1$ and $u_2 = 2$ are the positions of the first UV and IR renormalons, respectively. Let us further assume that the following L^2 condition holds:

$$\frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du\, |\widehat{B}_D(u)|^2 < \infty\,, \tag{2.14}$$

where $u_0 \in [u_1, u_2]$. Then the inverse Mellin transform of $\widehat{B}_D(u)$ can be defined as [34]

$$\widehat{w}_D(\tau) = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du\, \widehat{B}_D(u)\, \tau^{u-1}\,. \tag{2.15}$$

The condition (2.14) is satisfied by the Borel transform calculated in the large-β_0 limit, given in (2.10). In this case, the function $\widehat{w}_D(\tau)$ was first introduced in [15], where its physical interpretation as the distribution of the internal gluon virtualities in Feynman diagrams was pointed out. Relation (2.15) can be inverted to give

$$\widehat{B}_D(u) = \int\limits_0^\infty d\tau\, \widehat{w}_D(\tau)\, \tau^{-u}\,, \tag{2.16}$$

and the following completeness condition holds [34]:

$$\frac{1}{2\pi} \int\limits_{u_0-i\infty}^{u_0+i\infty} du\, |\widehat{B}_D(u)|^2 = \int\limits_0^\infty \frac{d\tau}{\tau}\, |\tau^{1-u_0}\, \widehat{w}_D(\tau)|^2\,. \tag{2.17}$$

The integral relation (2.16) defines the function $\widehat{B}_D(u)$ in a strip parallel to the imaginary axis with $u_1 < Re\, u < u_2$.

The distribution function $\widehat{w}_D(\tau)$ can be calculated from (2.15) by closing the integration contour along a semi-circle at infinity in the u plane and applying the theorem of residues for the singularities of $\widehat{B}_D(u)$ located inside the integration domain. For $\tau < 1$ the contribution from the semi-circle at infinity vanishes if the contour is closed in the right half of the u plane, while for $\tau > 1$ the contour must be closed in the left half plane. One thus obtains different expressions for the distribution function depending on whether $\tau < 1$ or $\tau > 1$, which we shall denote by $\widehat{w}_D^{(<)}(\tau)$ and $\widehat{w}_D^{(>)}(\tau)$, respectively. By Cauchy's theorem, these functions are given by the contribution of the residues of the IR and UV renormalons, respectively. For instance, $\widehat{w}_D^{(<)}(\tau)$ has the expression

$$\widehat{w}_D^{(<)}(\tau) = \frac{1}{2\pi i} \int_{\mathcal{C}_+} du\, \widehat{B}_D(u)\, \tau^{u-1} - \frac{1}{2\pi i} \int_{\mathcal{C}_-} du\, \widehat{B}_D(u)\, \tau^{u-1}\,, \tag{2.18}$$

where the contours $\mathcal{C}_+$ and $\mathcal{C}_-$ are lines slightly above and below the positive real axis, respectively. A similar expression applies for the function $\widehat{w}_D^{(>)}(\tau)$, where the integration lines are now parallel to the negative real axis. In the large-β_0 limit, these functions are given by [15]

$$\begin{aligned}
\widehat{w}_D^{(<)}(\tau) &= \frac{32}{3} \left\{ \tau \left(\frac{7}{4} - \ln\tau \right) + (1+\tau) \Big[L_2(-\tau) + \ln\tau \ln(1+\tau) \Big] \right\}\,, \\
\widehat{w}_D^{(>)}(\tau) &= \frac{32}{3} \left\{ 1 + \ln\tau + \left(\frac{3}{4} + \frac{1}{2} \ln\tau \right) \frac{1}{\tau} + (1+\tau) \Big[L_2(-\tau^{-1}) - \ln\tau \ln(1+\tau^{-1}) \Big] \right\}\,,
\end{aligned} \tag{2.19}$$

where $L_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$ is the dilogarithm. The above expressions are analytic in the complex τ plane, with no singularities other than branch cuts along the negative real axis. Together they define a function $\widehat{w}_D(\tau)$ that is piece-wise analytic in the cut τ plane, with different functional expressions for $|\tau| < 1$ and $|\tau| > 1$.

The fact that the function $\widehat{w}_D(\tau)$ is piece-wise analytic will have interesting consequences in our analysis below. One might ask whether these analytic properties are valid in general, i.e., beyond the large-β_0 limit. It is important in this context that we consider the analytic continuations of the function $\widehat{w}_D^{(<)}(\tau)$ defined in (2.18) and of the function $\widehat{w}_D^{(>)}(\tau)$ defined in a similar way, and not the original definition (2.15). The equivalence between the two results is valid only for real τ. By writing $\tau^u = \exp(u \ln \tau)$, and noticing that the residues of the renormalon singularities are real, we obtain from (2.18) real values for positive τ. However, a branch point at $\tau = 0$ and a cut at negative τ may appear due to the logarithm. In general, the singularities of the Borel transform are expected to be branch points (rather than poles) located on the real u axis. The inverse Mellin transforms of functions with branch points have a similar structure as those of functions with pole singularities [37]: they are piece-wise analytic functions, composed of two pieces analytic in the domains $|\tau| < 1$ and $|\tau| > 1$, with a cut along the negative real axis. Therefore, this appears to be the most general analytic structure for the function $\widehat{w}_D(\tau)$, assuming that the asymptotic behaviour of the Borel transform is in accord with the norm condition (2.14).

The application of the above procedure to obtaining the distribution function for the polarization amplitude itself requires some care due to the additional pole at $u = 0$ of the Borel transform $\widehat{B}_\Pi(u)$. An expression consistent with the definitions (2.2) and (2.3) is obtained if we define the inverse Mellin transform $\widehat{w}_\Pi(\tau)$ in terms of an integral along the line $Re\, u = u_0$ with $u_0 > 0$. Then, as proven in the Appendix, the result is

$$\widehat{w}_\Pi(\tau) = \frac{1}{\tau} \int_0^\tau dx\, \widehat{w}_D(x)\,. \tag{2.20}$$

In the large-β_0 limit, the explicit expression for this function is [15]

$$\widehat{w}_\Pi(\tau) = \frac{16}{3} \left\{ 1 - \ln x + \frac{x}{2}(5 - 3\ln x) + \frac{(1+x)^2}{x} \Big[L_2(-x) + \ln x \ln(1+x) \Big] \right\}, \tag{2.21}$$

where $x = \tau$ if $\tau < 1$, and $x = 1/\tau$ if $\tau > 1$. It follows that $\widehat{w}_\Pi^{(>)}(\tau) = \widehat{w}_\Pi^{(<)}(1/\tau)$.

Consider now the case of Minkowskian quantities such as the spectral moments M_k. In this case, the presence of the factor $\sin \pi u$ in the Borel transforms in (2.13) affects their asymptotic behaviour for large u and invalidates the L^2 condition (2.14). Therefore, in general an inverse Mellin transform defined as in (2.15) does not exist for Minkowskian quantities. This observation suggests to extract the sine factor from the Borel transform and consider the inverse Mellin transform of the remaining expression. Actually, such a definition can be considered not only for Minkowskian quantities but for any Borel transform $\widehat{B}_X(u)$ [14]. We define a function

$$\widehat{b}_X(u) = \frac{\widehat{B}_X(u)}{\sin \pi u} \tag{2.22}$$

and assume that it obeys the L^2 condition

$$\frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du\, |\widehat{b}_X(u)|^2 = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du \left| \frac{\widehat{B}_X(u)}{\sin \pi u} \right|^2 < \infty\,, \tag{2.23}$$

where $u_0 \neq 0$. In the case of Minkowskian quantities, the sine function in the denominator of (2.23) compensates the factor of $\sin \pi u$ appearing in the Borel transform. (In the case of Euclidean quantities the sine factor brings an additional improvement in the convergence of the integrals.) Provided the condition (2.23) is satisfied, we can then define the inverse Mellin transform $\widehat{W}_X(\tau)$ as

$$\widehat{W}_X(\tau) = \frac{1}{2\pi i} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} du\, \hat{b}_X(u)\, \tau^{u-1} = \frac{1}{2\pi i} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} du\, \frac{\widehat{B}_X(u)}{\sin \pi u}\, \tau^{u-1} \,. \tag{2.24}$$

This definition was introduced in a more physical context in [18]. Relation (2.24) can be inverted to give

$$\hat{b}_X(u) = \int\limits_0^\infty d\tau\, \widehat{W}_X(\tau)\, \tau^{-u} \,, \tag{2.25}$$

which together with (2.22) defines the Borel function $\widehat{B}_X(u)$ along a line $Re\, u = u_0 > 0$, where u_0 can be taken arbitrarily small. Of course, if the Borel transform $\widehat{B}_X(u)$ itself satisfies an L^2 condition like in (2.14), then also the normal inverse Mellin transform $\widehat{w}_X(\tau)$ defined as in (2.15) exists. In this case, one can prove the relation [16, 18]

$$\widehat{W}_X(\tau) = \frac{1}{\pi} \int\limits_0^\infty dx\, \frac{\widehat{w}_X(x)}{x + \tau} \,, \tag{2.26}$$

which we derive in the Appendix using Parseval's theorem. The representation (2.26) shows that, unlike the distribution functions $\widehat{w}_D(\tau)$ and $\widehat{w}_\Pi(\tau)$, which are piece-wise analytic in the τ plane, the inverse Mellin transforms $\widehat{W}_D(\tau)$ and $\widehat{W}_\Pi(\tau)$ are real analytic functions in the entire complex τ plane cut along the real negative axis. (The explicit expression for the function $\widehat{W}_D(\tau)$ can be found in the Appendix of [16].) On the other hand, for the Minkowskian quantities the distribution functions $\widehat{W}_X(\tau)$ are piece-wise analytic, with two pieces defined for $|\tau| < 1$ and $|\tau| > 1$ [18].

3. Borel sums in the complex momentum plane

Borel-summed expressions for the polarization function at both space-like and time-like momenta in terms of the distribution functions $\widehat{w}_D(\tau)$ and $\widehat{W}_D(\tau)$ were derived in [15] and [18], respectively. In what follows, we will generalize the techniques used in these works to the calculation of the polarization function in the complex momentum plane. For definiteness, we adopt as a regularization prescription for the ill-defined integral (2.7) the so-called "principal value", which amounts to taking one half of the sum of the integrals along two parallel lines slightly above and below the real axes, denoted by $\mathcal{C}_\pm$ (the same integration lines appeared previously in (2.18)). We thus define

$$D(s) = 1 + \frac{1}{2} \left[d_+(s) + d_-(s) \right] , \tag{3.1}$$

where

$$d_{\pm}(s) = \frac{1}{\beta_0} \int_{\mathcal{C}_\pm} du \left(\frac{-s}{\Lambda_V^2} \right)^{-u} \widehat{B}_D(u)\,, \tag{3.2}$$

and s is taken to lie outside the Landau region, $|s| > \Lambda_V^2$, so that the integrals are convergent. The prescription (3.1) is a generalization of the Cauchy principal value for simple poles, which was adopted in [33] as the only choice giving a real value for the Borel-summed amplitude when the coupling constant is real. In the renormalization-group improved expansion, this means that $D(s)$ is real along the space-like s axis outside the Landau region. This is consistent with the dispersion relation (2.4), which follows from general causality and unitarity requirements. As we shall see, other prescriptions violate these requirements.

Our aim is to express the integrals (3.2) in terms of the distribution functions defined in the previous section. To this end, we must pass from the integrals along the contours $\mathcal{C}_\pm$ to integrals along a line parallel to the imaginary axis, where the representation (2.16) is valid. This can be achieved by rotating the integration contour from the real to the imaginary axis, provided the contour at infinity does not contribute. In each individual case, one must establish that this condition is satisfied. Let us consider first a point in the upper half of the momentum plane, for which $s = |s|\,e^{i\phi}$ and $-s = |s|\,e^{i(\phi-\pi)}$ with a phase $0 < \phi < \pi$. Taking $u = \mathcal{R}\,e^{i\theta}$ on a large semi-circle of radius $\mathcal{R}$, the relevant exponential appearing in the integrals (3.2) is

$$\exp\left\{ -\mathcal{R} \left[\ln \frac{|s|}{\Lambda_V^2} \cos\theta + (\pi - \phi)\sin\theta \right] \right\}. \tag{3.3}$$

For $|s| > \Lambda_V^2$, the exponential is negligible at large $\mathcal{R}$ for $\cos\theta > 0$ and $\sin\theta > 0$, i.e., for the first quadrant of the complex u plane. Assuming that $\widehat{B}_D(u)$ increases slower than any exponential [18], the integration contour defining $d_+(s)$ can be rotated to the positive imaginary axis, where the representation (2.16) is valid. This leads to the double integral

$$d_+(s) = \frac{1}{\beta_0} \int\limits_0^{i\infty} du \int\limits_0^{\infty} d\tau\, \widehat{w}_D(\tau) \exp\left[-u \left(\ln \frac{\tau |s|}{\Lambda_V^2} + i(\phi - \pi) \right) \right]. \tag{3.4}$$

The order of integrations over τ and u can be interchanged, since for negative $(\phi - \pi)$ the integral over u is convergent. Performing this integral yields

$$d_+(s) = \frac{1}{\beta_0} \int\limits_0^{\infty} d\tau\, \frac{\widehat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)}\,. \tag{3.5}$$

Consider now the evaluation of the function $d_-(s)$ given by the integral along the contour $\mathcal{C}_-$ below the real axis. Naively, one might think to rotate the integration contour to the negative imaginary axis without crossing any singularities. However, this rotation is not allowed, because along the corresponding quarter-circle $\sin\theta < 0$, and the exponent (3.3) does not vanish at infinity for $\phi < \pi$. The way out is to perform again a rotation to the positive imaginary u axis, for which the contribution of the circle at infinity vanishes. But in this rotation the contour crosses the positive real axis, and hence we must pick up the contributions of the IR renormalon singularities located along this line. This is easily achieved by comparing the expression (2.18) for the function $\widehat{w}_D^{(<)}(\tau)$ with the definition (3.2) of the functions $d_\pm(s)$. It follows that $d_-(s)$ can be expressed in terms of $d_+(s)$ as

$$d_-(s) = d_+(s) - \frac{2\pi i}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\widehat{w}_D^{(<)}(-\Lambda_V^2/s)\,. \tag{3.6}$$

The relations (3.5) and (3.6) completely specify the function $D(s)$ in the upper half of the momentum plane. Using the same method, the Adler function can be calculated in the lower half plane, which corresponds to taking $\pi < \phi < 2\pi$. In this case, the integral along $\mathcal{C}_-$ can be calculated by rotating the contour to the negative imaginary u axis, while for the integration along $\mathcal{C}_+$ one must pass across the real axis. Combining the results, we obtain the following expression for the Adler function in the complex momentum plane:

$$D^{(\pm)}(s) = 1 + \frac{1}{\beta_0}\int\limits_0^\infty d\tau\,\frac{\widehat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)} \mp \frac{i\pi}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\widehat{w}_D^{(<)}(-\Lambda_V^2/s)\,, \tag{3.7}$$

where the superscript "$\pm$" in parenthesis refers to the sign of $Im\, s$. The corresponding expression for the polarization function $\Pi(s)$ can be obtained by inserting the above result into the definition (2.3). This gives

$$\Pi^{(\pm)}(s) = k - \ln\left(\frac{-s}{\Lambda_V^2}\right) - \frac{1}{\beta_0}\int\limits_0^\infty d\tau\,\widehat{w}_D(\tau)\ln\ln\left(-\frac{\tau s}{\Lambda_V^2}\right) \mp \frac{i\pi}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\widehat{w}_\Pi^{(<)}(-\Lambda_V^2/s)\,, \tag{3.8}$$

with $\widehat{w}_\Pi(\tau)$ as defined in (2.20). Note that the last terms in (3.7) and (3.8) involve the analytic continuation of the Mellin transforms $\widehat{w}_D(\tau)$ and $\widehat{w}_\Pi(\tau)$ from the real positive axis, where they have been calculated, to arbitrary complex arguments.

Before discussing in the next section the analyticity properties of these results, we apply the same technique to the principal-value Borel summation of Minkowskian quantities. Let us generically denote a Minkowskian quantity by $R(s_0)$, with s_0 some fixed scale, and write its Laplace integral with the principal-value prescription as

$$R(s_0) = R_0 + \frac{1}{2}\Big[r_+(s_0) + r_-(s_0)\Big]\,, \tag{3.9}$$

where R_0 is a constant, and

$$r_\pm(s_0) = \frac{1}{2i\beta_0}\int_{\mathcal{C}_\pm} du\,\widehat{b}_R(u)\left\{\exp\left[-u\ln\left(\frac{s_0}{\Lambda_V^2}\right) + i\pi u\right] - \exp\left[-u\ln\left(\frac{s_0}{\Lambda_V^2}\right) - i\pi u\right]\right\}\,. \tag{3.10}$$

We have expressed the Borel transform $\widehat{B}_R(u)$ in terms of $\widehat{b}_R(u)$ as in (2.22) and combined the two exponentials arising from the factor $\sin\pi u$ with the exponential in the Laplace integral. We recall that $s_0 > \Lambda_V^2$. The procedure described in detail for complex s can be now applied in a straightforward way. For $\mathcal{C}_+$ ($\mathcal{C}_-$) one can rotate the integral of the first (second) term in (3.10) towards the positive (negative) imaginary axis, where the integral representation (2.25) of $\widehat{b}_R(u)$ in terms of $\widehat{W}_R(\tau)$ can be used. For the remaining terms, i.e., the second (first) term in the integral along $\mathcal{C}_+$ ($\mathcal{C}_-$), the simple rotation towards the imaginary axis cannot be performed, since the corresponding terms $\pm i\pi u$ in the exponential blow up along the quadrants of the circle. As explained before, we must first cross the real axis in the u plane and then make a rotation towards the nearest imaginary semi-axis. But in crossing the real axis we encounter the IR renormalons, whose contribution must be added using an analog of (3.6). In the present case, the relation is

$$\begin{aligned}&\int_{\mathcal{C}_+} du\,\widehat{b}_R(u)\left\{\exp\left[-u\ln\left(\frac{s_0}{\Lambda_V^2}\right)\pm i\pi u\right]\right\}\\ &= \int_{\mathcal{C}_-} du\,\widehat{b}_R(u)\left\{\exp\left[-u\ln\left(\frac{s_0}{\Lambda_V^2}\right)\pm i\pi u\right]\right\} + 2\pi i\tau_\pm \widehat{W}_R^{(<)}(\tau_\pm)\,,\end{aligned} \tag{3.11}$$

where $\tau_\pm = -\Lambda_V^2/s_0 \pm i\epsilon$. Using this relation, we find after a straightforward calculation

$$R(s_0) = R_0 + \frac{\pi}{\beta_0}\int_0^\infty d\tau\,\frac{\widehat{W}_R(\tau)}{\ln^2(\tau s_0/\Lambda_V^2)+\pi^2} + \frac{\pi}{2\beta_0}\frac{\Lambda_V^2}{s_0}\left[\widehat{W}_R^{(<)}(\tau_+) + \widehat{W}_R^{(<)}(\tau_-)\right]. \tag{3.12}$$

Notice that the last two terms involve the values of the function $\widehat{W}_R^{(<)}(\tau)$ for real negative values of the argument, where this function has a cut. In the two terms the cut is approached from opposite directions, and the sum gives twice the real part of the function $\widehat{W}_R^{(<)}(\tau)$ at $\tau = -\Lambda_V^2/s_0$. It is convenient to rewrite the above result using an integration by parts, yielding

$$R(s_0) = R_0 + \frac{1}{\beta_0}\int_0^\infty d\tau\, I_R(\tau)\,\arctan\left[\frac{\pi}{\ln(\tau s_0/\Lambda_V^2)}\right] + \frac{\pi}{\beta_0}\,Re\int_{-\Lambda_V^2/s_0}^{\Lambda_V^2/s_0} d\tau\, I_R(\tau - i\epsilon)\,, \tag{3.13}$$

where

$$I_R(\tau) = \frac{d}{d\tau}\left[\tau\,\widehat{W}_R(\tau)\right]. \tag{3.14}$$

The integral representations (3.12) and (3.13) were derived previously in [18] by means of a different technique of treating the integrals (3.10). Specifically, in order to avoid the crossing of the real axis in the rotation of the contours $\mathcal{C}_\pm$ towards the imaginary axis, these authors use formally the integral representation (2.25) of $\widehat{b}_R(u)$ outside its range of validity. The double integral thus obtained is evaluated by performing first a suitable rotation of the integration line in the τ plane, and then a rotation of the integration contour in the u plane. The formal application of this procedure leads to an expression identical to (3.12).

For our further discussion, it will be useful to have an explicit expression for the functions $I_{M_k}(\tau)$ entering the integral representation (3.13) in the particular case of the Borel sum of the spectral moments M_k. These functions are given by [20][2]

$$I_{M_k}(\tau) = \frac{k+1}{\pi} \int\limits_0^1 dx\, x^{k-1}\, \widehat{w}_D(\tau/x) = \frac{k+1}{\pi}\, \tau^k \int\limits_\tau^\infty dz\, z^{-k-1}\, \widehat{w}_D(z)\,, \qquad \tau > 0\,. \quad (3.15)$$

We stress that from the derivation of the result (3.13) it follows that for negative values of τ the functions $I_{M_k}(\tau)$ must be obtained from the analytic continuation of the expressions derived from for τ real and positive. Note that for $\tau < 0$ these functions are not the same as the functions

$$I'_{M_k}(\tau) = \frac{k+1}{\pi}\, (-\tau)^k \int\limits_{-\tau}^\infty dz\, z^{-k-1}\, \widehat{w}_D(-z)\,, \qquad \tau < 0 \qquad (3.16)$$

one would obtain using the analytic continuation of the function $\widehat{w}_D(\tau/x)$ under the first integral in (3.15). This observation will become important in the next section.

4. Analyticity properties of the Borel-summed functions

Let us now investigate in more detail the momentum-plane analyticity properties of the Adler function and of the polarization function calculated in the previous section. We start with the Borel sum of the function $D(s)$. From (3.7), it is apparent that its s dependence is isolated in two terms having a different origin. The first term can be regarded as an average over gluon virtualities of the one-loop expressions in (2.6) [15]. It is obtained using the integral representation of the Borel function in terms of the inverse Mellin transform along the imaginary axis in the u plane. The second term is generated by the residues of the IR renormalons when crossing the real axis to pass from one of the contours $\mathcal{C}_\pm$ to the other. As we will see, the fact that this term contains the piece $\widehat{w}_D^{(<)}(\tau)$ of the distribution function has important consequences on the analyticity properties.

As shown in (3.7), we obtained two different expressions for the Adler function valid in the upper and lower half of the momentum plane. Using general theorems about the analyticity of functions represented by integrals [39, 40] and the fact that $\widehat{w}_D^{(<)}(\tau)$ is holomorphic for complex values of τ, we find that the functions $D^{(\pm)}(s)$ are holomorphic for complex values of s outside the real axis, in the upper and lower half planes, respectively. We will now show that they represent a unique function

[2] In this reference, results are given for the functions $W_k(\tau) = \pi\tau\, I_{M_k}(\tau)$.

analytic in the cut momentum plane. To this end, we apply the Schwarz reflection principle. The functions $D^{(\pm)}(s)$ are related to each other by the reality condition $[D^{(+)}(s^*)]^* = D^{(-)}(s)$, since $w_D^{(<)}(\tau)$ is a real analytic function, which enters the expressions for $D^{(+)}(s)$ and $D^{(-)}(s)$ with imaginary coefficients of opposite sign. Assume now that s is approaching the real axis from the upper half plane. Using expression (3.7) together with the relation

$$Im\left[\frac{1}{\ln[-(x+i\epsilon)]}\right] = \theta(x)\,\frac{\pi}{\ln^2(x)+\pi^2} + \pi\delta(x+1)\,, \tag{4.1}$$

we obtain on the time-like axis

$$Im\,D(s+i\epsilon) = \frac{\pi}{\beta_0}\left[\int\limits_0^\infty d\tau\,\frac{\widehat{w}_D(\tau)}{\ln^2(\tau s/\Lambda_V^2)+\pi^2} + \frac{\Lambda_V^2}{s}\,Re\,\widehat{w}_D^{(<)}(-\Lambda_V^2/s)\right], \qquad s>0\,, \tag{4.2}$$

while on the space-like axis

$$Im\,D(s+i\epsilon) = \frac{\pi}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\left[\widehat{w}_D(-\Lambda_V^2/s) - \widehat{w}_D^{(<)}(-\Lambda_V^2/s)\right], \qquad s<0\,. \tag{4.3}$$

We recall that we have assumed that s lies outside the Landau region, i.e., $|s| > \Lambda_V^2$. In this case, the argument of the functions $\widehat{w}_D(\tau)$ and $\widehat{w}_D^{(<)}(\tau)$ appearing in (4.3) is less than one. But for such values the two functions coincide, and the two terms in (4.3) compensate each other. Therefore, the imaginary part of the Borel-summed Adler function $D(s)$ vanishes for space-like momenta outside the Landau region:

$$Im\,D(s+i\epsilon) = 0\,, \qquad s < -\Lambda_V^2\,. \tag{4.4}$$

The same result is obtained when approaching the real negative axis from below. Therefore, in the deep Euclidian region $-s > \Lambda_V^2$ both expressions in (3.7) lead to a real value given by

$$D(s) = 1 + \frac{1}{\beta_0}\,Re\int\limits_0^\infty d\tau\,\frac{\widehat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)}\,, \qquad s < -\Lambda_V^2\,. \tag{4.5}$$

By the Schwarz reflexion principle, the two expressions $D^{(\pm)}(s)$ in (3.7) define a single function $D(s)$ real analytic in the momentum plane, with no cut for $s < -\Lambda_V^2$.

Before proceeding, it is worth emphasizing that the vanishing of the imaginary part on the space-like s axis outside the Landau region is not a generic feature of the Borel sum for the Adler function, but is specific to the principal-value prescription adopted above. It does not happen if one takes a different prescription for avoiding the singularities of the Laplace integral. For instance, taking the integral along the contour $\mathcal{C}_+$ as a definition of the Borel integral we would obtain instead of (3.7) the expressions

$$\begin{aligned} D_+^{(+)}(s) &= 1 + \frac{1}{\beta_0}\int\limits_0^\infty d\tau\,\frac{\widehat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)}\,, \\ D_+^{(-)}(s) &= 1 + \frac{1}{\beta_0}\int\limits_0^\infty d\tau\,\frac{\widehat{w}_D(\tau)}{\ln(-\tau s/\Lambda_V^2)} + \frac{2\pi i}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\widehat{w}_D^{(<)}(-\Lambda_V^2/s)\,, \end{aligned} \tag{4.6}$$

where the subscript "+" indicates the regularization prescription. It is obvious that these expressions do not satisfy the condition of real analyticity $D^*(s) = D(s^*)$. In particular, using the relation (4.1) we see that the two expressions in (4.6) have non-zero imaginary parts along the whole space-like axis $s < 0$. (These imaginary parts coincide, while the real parts exhibit a discontinuity across the real axis.) On the other hand, for real analytic functions the discontinuity is due to the imaginary part, since in this case $D(s + i\epsilon) - D(s - i\epsilon) = 2i\,Im\,D(s + i\epsilon)$. By unitarity, the discontinuity of the exact function $D(s)$ is non-zero only above the threshold for hadron production, and below this threshold the exact function must be real.[3] The expressions in (4.6) are in conflict with the above requirements, and in our opinion this fact is an argument in favour of the principal-value prescription.

The explicit expression for $D(s)$ derived in (3.7) for $|s| > \Lambda_V^2$ can be analytically continued inside the circle $|s| < \Lambda_V^2$. It is interesting to make this continuation in order to explore what happens, after Borel summation, with the Landau pole present in the fixed-order perturbative expression (2.6). We find that for complex values of s outside the real axis the resulting expression is holomorphic. The discontinuities along the real axis can be analysed using (4.2) and (4.3). On the time-like axis, we find a rather complicated behaviour besides the expected unitarity branch point at $s = 0$ produced by the logarithm. Indeed, the spectral function at $s < \Lambda_V^2$ involves the values of the function $\widehat{w}_D^{(<)}(\tau)$ extended analytically to the range $|\tau| > 1$, where there may be additional singularities (see, e.g., the explicit expressions (2.19) valid in the large-β_0 limit). Singularities are also present along the space-like region $-\Lambda_V^2 < s < 0$. Indeed, in this case the argument of the functions $\widehat{w}_D(\tau)$ and $\widehat{w}_D^{(<)}(\tau)$ in (4.3) is $\Lambda_V^2/|s| > 1$. Therefore, the last two terms in (4.3) do not compensate each other anymore, and hence the analytic continuation of the expression for $D(s)$ derived above has a cut along the low-energy Euclidian region with the discontinuity

$$Im\,D(s+i\epsilon) = \frac{\pi}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\left[\widehat{w}_D^{(>)}(-\Lambda_V^2/s) - \widehat{w}_D^{(<)}(-\Lambda_V^2/s)\right]\,, \quad -\Lambda_V^2 < s < 0\,. \tag{4.7}$$

The appearance of this cut is explicitely related to the fact that the inverse Mellin transform of the Borel function is a piece-wise analytic function. Note that even if

[3]These properties, which follow from the general principles of field theory, are explicitely incorporated in the dispersion relation (2.4), due to the fact that the spectral function $Im\,\Pi(s)$ is real.

IR renormalons were absent and the theory was Borel summable, the Landau cut would persist. Indeed, in this case $d_+(s) = d_-(s)$, and from (3.6) it follows that the function $\widehat{w}_D^{(<)}(\tau)$ vanishes identically. But in (4.7) we are then left with the term $\widehat{w}_D^{(>)}(\tau)$, which receives contributions from the UV renormalons.

The presence of the Landau cut (4.7) is intimately related with the asymptotic behaviour of the Borel transform. It is straightforward to show that if the Borel transform vanishes sufficiently fast at infinity in the complex u plane — more precisely, if not only $\widehat{B}_D(u)$ but also the product $\widehat{B}_D(u)\,\sin\pi u$ are L^2 integrable functions in the sense of (2.14) — then the Landau cut is absent. Indeed, in this case a reasoning similar to that presented in the discussion containing equations (2.22)–(2.26) leads to the conclusion that the inverse Mellin transform $\widehat{w}_D(\tau)$ satisfies a dispersion representation and is therefore analytic in the complex τ plane cut along the negative axis. Then the two pieces $\widehat{w}_D^{(<)}(\tau)$ and $\widehat{w}_D^{(>)}(\tau)$ coincide, and the imaginary part (4.7) vanishes identically. However, the strong decrease of the Borel transform ensuring the absence of the Landau cut is not observed in the large-β_0 limit, and it is thus rather improbable that it should happen in the physical case.

As a side remark, let us indicate what happens if the effective coupling constant is modified in the IR domain so as not to contain a Landau pole. To study this case we make the replacement

$$-\frac{\Lambda_V^2}{s} \;\to\; \tau_L \equiv \exp\left(-\frac{4\pi}{\beta_0\,\alpha_s(-s)}\right)\,, \tag{4.8}$$

where the coupling $\alpha_s(-s)$ is defined in the V scheme. Then the discontinuity (4.3) across the Landau cut may be written as

$$Im\,D(s+i\epsilon) = \frac{\pi}{\beta_0}\,\tau_L\left[\widehat{w}_D(\tau_L) - \widehat{w}_D^{(<)}(\tau_L)\right]\,, \qquad s<0\,. \tag{4.9}$$

The coupling $\alpha_s(-s)$ may be defined by a dispersion integral [29] or simply by subtracting the pole at $s = -\Lambda_V^2$ by hand. As discussed in [18], this redefinition amounts to adding terms exponentially small in the coupling, which do not modify the perturbative expansion. The modified coupling is real and positive along the whole space-like region $s<0$, implying that τ_L defined in (4.8) is a positive number less than unity. As a consequence, the imaginary part of $D(s)$ vanishes along the space-like axis, since for $\tau_L<1$ the two terms in (4.9) compensate each other. Therefore, unphysical singularities do not appear in the Borel-summed expansion in powers of a regular coupling, if the principal-value prescription is adopted for treating the IR renormalons.[4]

[4]This argument cannot be considered a proof, since the very meaning of the Borel summation of a modified expansion with the Landau pole removed is not transparent. Indeed, the redefinition of the coupling itself may be seen as a partial summation of some higher-order effects, which must be separated from the terms taken into account in the Borel sum in order to avoid double counting.

JHEP03(1999)007

For completeness, we also present results for the polarization function $\Pi(s)$. The Borel-summed expression (3.8), obtained for $|s| > \Lambda_V^2$, can be analytically continued to the whole complex plane. We note that $\Pi(s)$ is holomorphic for complex values of s and satisfies the reality condition $\Pi(s^*) = \Pi^*(s)$. On the real axis, this function can have singularities manifested as discontinuities of the imaginary part. A straightforward calculation gives

$$\begin{aligned}
Im\,\Pi(s+i\epsilon) &= \pi + \frac{1}{\beta_0}\int\limits_0^\infty d\tau\,\widehat{w}_D(\tau)\arctan\left(\frac{\pi}{\ln(\tau s/\Lambda_V^2)}\right) + \\
&\quad + \frac{\pi}{\beta_0}\,Re\int\limits_{-\Lambda_V^2/s}^{\Lambda_V^2/s} d\tau\,\widehat{w}_D(\tau)\,, \qquad s>0\,, \\
Im\,\Pi(s+i\epsilon) &= \frac{\pi}{\beta_0}\left(-\frac{\Lambda_V^2}{s}\right)\left[\widehat{w}_\Pi(-\Lambda_V^2/s) - \widehat{w}_\Pi^{(<)}(-\Lambda_V^2/s)\right]\,, \qquad s<0\,.
\end{aligned} \tag{4.10}$$

Again, the second expression vanishes outside the Landau region, where the function $\widehat{w}_\Pi(\tau)$ coincides with $\widehat{w}_\Pi^{(<)}(\tau)$. However, inside the interval $-\Lambda_V^2 < s < 0$ the function $\Pi(s)$ has a non-zero imaginary part.

The final point we will discuss concerns the implications of the above results for the Borel summation of Minkowskian quantities such as the spectral moments M_k. In (3.13) we derived the principal-value Borel sum for these quantities, which we shall denote by $M_k^{\rm Borel}$. On the other hand, starting from the explicit expression (3.8) for the polarization function $\Pi(s)$ in the complex plane, we could calculate its moments also by performing the contour integral in the second relation in (2.11). Let us denote by $M_k^{\rm circle}$ the result of this procedure. We are now going to show that $M_k^{\rm Borel} = M_k^{\rm circle}$, which is equivalent to the statement that contour integration and Borel summation commute with each other. In order to compare the two resummed expressions for the moments, we pass from the integral along the circle $|s| = s_0$ to an integral along the real axis. Applying the Cauchy relation, and taking into account the singularities of the Borel sum of $\Pi(s)$ inside the circle, we write in the most general way

$$\begin{aligned}
M_k^{\rm circle} &\stackrel{\eta\to 0}{=} -\frac{k+1}{2\pi i\,s_0^{k+1}}\oint\limits_{|s|=\eta} ds\,s^k\,\Pi(s) + \\
&\quad + \frac{k+1}{\pi\,s_0^{k+1}}\left[\int\limits_{-s_0}^{-\eta} ds\,s^k\,Im\,\Pi(s+i\epsilon) + \int\limits_\eta^{s_0} ds\,s^k\,Im\,\Pi(s+i\epsilon)\right].
\end{aligned} \tag{4.11}$$

The point $s=0$ must be treated separately since the function $\widehat{w}_\Pi^{(<)}(-\Lambda_V^2/s)$ entering the expression for $\Pi(s)$ in (3.8) is singular at this point. (In the large-β_0 limit, we have $\widehat{w}_\Pi^{(<)}(\tau) \sim \tau\ln^2\tau$ for $\tau\to\infty$.) From this behaviour, it follows that the circle of radius η gives a non-vanishing contribution only for the first two moments. For $k \geq 2$, the only singularities are due to the discontinuity of the imaginary part of $\Pi(s)$ given above. As we discussed, this imaginary part vanishes for $s < -\Lambda_V^2$. By a straightforward calculation, using (4.10) and isolating the contribution of the region $-\Lambda_V^2 < s < \Lambda_V^2$, we write (4.11) as $M_k^{\rm circle} = M_k' + \delta_k$, where

$$\delta_k \overset{\eta\to 0}{=} -\frac{k+1}{2\pi i\, s_0^{k+1}} \oint\limits_{|s|=\eta} ds\, s^k\, \Pi(s) + \\ +\frac{k+1}{\beta_0\, s_0^{k+1}} \left[\int\limits_{-\Lambda_V^2}^{-\eta} ds\, s^k\, \Delta(-\Lambda_V^2/s) + \int\limits_{\eta}^{\Lambda_V^2} ds\, s^k\, Re\, \Delta(-\Lambda_V^2/s) \right], \tag{4.12}$$

with

$$\Delta(\tau) = \tau \left[\widehat{w}_\Pi^{(>)}(\tau) - \widehat{w}_\Pi^{(<)}(\tau) \right]. \tag{4.13}$$

The quantity M_k' has an expression that is formally identical with the Borel moments $M_k^{\rm Borel}$ in (3.13), but in which the functions $I_{M_k}(\tau)$ in (3.15) are replaced with the functions $I_{M_k}'(\tau)$ in (3.16) obtained using a different prescription for negative τ. In other words, we have $M_k' = M_k^{\rm Borel} - \delta_k'$ with

$$\delta_k' = \frac{\pi}{\beta_0}\, Re \int\limits_{-\Lambda_V^2/s_0}^{0} d\tau \left[I_{M_k}(\tau - i\epsilon) - I_{M_k}'(\tau - i\epsilon) \right]. \tag{4.14}$$

An explicit calculation in the large-β_0 limit shows that $\delta_k = \delta_k'$, and we believe this result is of general validity. From this observation, it follows that indeed $M_k^{\rm circle} = M_k^{\rm Borel}$, which is in agreement with the findings of [18]. Note that the arguments given above show that the equivalence of the two resummation procedures results from a subtle compensation of the contribution of the singularities of $\Pi(s)$ along the Landau region with an additional term arising from the analytic continuation of a piece-wise analytic function.

5. Conclusions

We have derived a compact expression for the Borel-summed Adler function and the polarization amplitude in the complex momentum plane in terms of the inverse Mellin transform of the corresponding Borel functions. At present, these inverse Mellin transforms (or distribution functions) are known only in the large-β_0 limit. However, the expressions derived in the present work are useful for investigating the analyticity properties in the complex momentum plane, and therefore some conclusions can be drawn even if the distribution functions are not known exactly.

JHEP03(1999)007

Our main results are the expressions for the Borel sums of the functions $D(s)$ and $\Pi(s)$ given in (3.7) and (3.8), respectively. They have been derived using a generalized principal-value prescription, which leads to real values of the functions along the space-like s axis outside the Landau region, in accordance with general analyticity requirements imposed by causality and unitarity. The expressions derived in the momentum plane outside the Landau region admit an explicit analytical continuation to the region $|s| < \Lambda_V^2$, revealing Landau singularities more complicated than those present in fixed-order perturbation theory. We have shown that the discontinuity across the Landau cut is connected with the piece-wise character of the inverse Mellin transform, which is in turn correlated with the asymptotic behaviour of the Borel transform. We have also presented a condition on the asymptotic behaviour of the Borel transform that would ensure the vanishing of the Landau cut. This condition is, however, not satisfied in the large-β_0 limit, and it will most likely not be satisfied for real QCD. Therefore, we do expect the Landau cut to be a general feature of Borel-summed perturbation theory in QCD.

With the explicit expressions derived in the complex momentum plane, we have checked that the spectral moments defined by contour integrals of the Borel sum of the correlator $\Pi(s)$ coincide with the "standard" Borel moments considered previously in [17]–[20]. This equality is consistent with the complicated singularity structure of the polarization function discovered in this work, providing an indirect check of our results.

We have shown that the unphysical discontinuity across the Landau cut would persist in the fictitious case of a Borel-summable perturbation series. Thus, the Landau singularities in the momentum plane and the IR renormalons are not directly related to each other. A similar conclusion was formulated in the context of a specific model in [22]. The unphysical cut is a manifestation of the incompleteness of any perturbative approximation to a physical hadronic quantity. On the other hand, our results suggest that, if the running coupling is regularized by a dispersion relation, then the analyticity properties of fixed-order perturbation theory are preserved by Borel summation. Indeed, in this case the discontinuity across the Landau cut vanishes even if IR renormalons are present. This result holds under conservative assumptions about the properties of the inverse Mellin transform and thus seems to have a general validity.

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JHEP03(1999)007

A. Relations between distribution functions

Here we indicate a simple method for calculating the distribution functions of various observables using the relations between their Borel transforms. To this end, we use Parseval's theorem for the Mellin transform [34], which can be stated as follows: if $\mathcal{F}(u)$ and $\mathcal{G}(u)$ satisfy the L^2 conditions

$$\frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} du\,|\mathcal{F}(u)|^2 < \infty\,, \qquad \frac{1}{2\pi i}\int_{1-u_0-i\infty}^{1-u_0+i\infty} du\,|\mathcal{G}(u)|^2 < \infty\,, \tag{A.1}$$

where $0 < u_0 < 1$, then

$$\frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} du\,\mathcal{F}(u)\,\mathcal{G}(1-u) = \int_0^\infty dx\,f(x)\,g(x)\,, \tag{A.2}$$

where $f(x)$ and $g(x)$ are the inverse Mellin transforms

$$f(x) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} du\,\mathcal{F}(u)\,x^{u-1}\,, \qquad g(x) = \frac{1}{2\pi i}\int_{1-u_0-i\infty}^{1-u_0+\infty} du\,\mathcal{G}(u)\,x^{u-1}\,. \tag{A.3}$$

As a first application of this result, we use it to prove relation (2.26). We start with relation (2.24) and notice that the function $\widehat{W}_X(\tau)$ can be written as

$$\widehat{W}_X(\tau) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} du\,\mathcal{F}(u)\,\mathcal{G}(1-u)\,, \tag{A.4}$$

where

$$\mathcal{F}(u) = \widehat{B}_X(u)\,, \qquad \mathcal{G}(u) = \frac{\tau^{-u}}{\sin\pi(1-u)}\,. \tag{A.5}$$

It then follows that $f(x) = \widehat{w}_X(x)$, and inserting the definition (A.5) of the function $\mathcal{G}(u)$ into (A.3) and making the change of variables $s = 1-t$ we find

$$g(x) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+\infty} ds\,\frac{\tau^{s-1}x^{-s}}{\sin\pi s}\,. \tag{A.6}$$

This integral can be calculated closing the contour in the complex s plane and applying the theorem of residues. For $Re\,(\tau/s) > 0$ we close the contour in the left half plane $Re\,s < 0$, while for $Re\,(\tau/s) < 0$ we close it in the right half plane $Re\,s > 0$. In each case, we pick up the contributions of the relevant poles situated at $s = \pm n$ with the residues

$$\frac{(-1)^n}{\pi\tau}\left(\frac{\tau}{x}\right)^n. \tag{A.7}$$

The contributions of the semi-circles at infinity vanish due to the exponentials in the sine function, even for complex values of $\tau/x = |\tau/x|\exp(\pm i\psi)$ with $|\psi| < \pi$. Therefore, we obtain

$$g(x) = \frac{1}{\pi\tau}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\tau}{x}\right)^n = \frac{1}{\pi(x+\tau)}\,. \tag{A.8}$$

Inserting the expressions for the functions $f(x)$ and $g(x)$ into the right-hand side of (A.2), and using the definition of $\widehat{W}_X(\tau)$, we obtain

$$\widehat{W}_X(\tau) = \frac{1}{\pi}\int_0^\infty dx\,\frac{\widehat{w}_X(x)}{x+\tau}\,, \tag{A.9}$$

where τ can take arbitrary values, except for real negatives.

As a second example, we derive relation (2.20), starting from

$$\widehat{w}_\Pi(\tau) = \frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} du\,\frac{\widehat{B}_D(u)}{u}\,\tau^{u-1}\,, \tag{A.10}$$

where we have used (2.9). The above result can be rewritten as in (A.4), with the identification

$$\mathcal{F}(u) = \widehat{B}_D(u)\,, \qquad \mathcal{G}(u) = \frac{\tau^{-u}}{(1-u)}\,. \tag{A.11}$$

A straightforward calculation gives the inverse Mellin transforms

$$f(x) = \widehat{w}_D(x)\,, \qquad g(x) = \frac{1}{\tau}\,\theta\left(1-\frac{x}{\tau}\right)\,. \tag{A.12}$$

Inserting these functions into the right-hand side of Parseval's theorem (A.2), we obtain relation (2.20).

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JHEP03(1999)007